Chapter 4 One Dimensional Kinematics

Chapter 4 One Dimensional Kinematics

In the first place, what do we mean by time and space? It turns out that these deep philosophical questions have to be analyzed very carefully in physics, and this is not easy to do. The theory of relativity shows that our ideas of space and time are not as simple as one might imagine at first sight. However, for our present purposes, for the accuracy that we need at first, we need not be very careful about defining things precisely. Perhaps you say, "That's a terrible thing—I learned that in science we have to define everything precisely." We cannot define anything precisely! If we attempt to, we get into that paralysis of thought that comes to philosophers, who sit opposite each other, one saying to the other, "You don't know what you are talking about!" The second one says. "What do you mean by know? What do you mean by talking? What do you mean by you?", and so on. In order to be able to talk constructively, we just have to agree that we are talking roughly about the same thing. You know as much about time as you need for the present, but remember that there are some subtleties that have to be discussed; we shall discuss them later.[1]

Richard Feynman

4.1 Introduction

Kinematics is the mathematical description of motion. The term is derived from the Greek word *kinema,* meaning movement. In order to quantify motion, a mathematical coordinate system, called a *reference frame*, is used to describe space and time. Once a reference frame has been chosen, we can introduce the physical concepts of position, velocity and acceleration in a mathematically precise manner. Figure 4.1 shows a Cartesian coordinate system in one dimension with unit vector $\hat{\mathbf{i}}$ pointing in the direction of increasing x-coordinate.

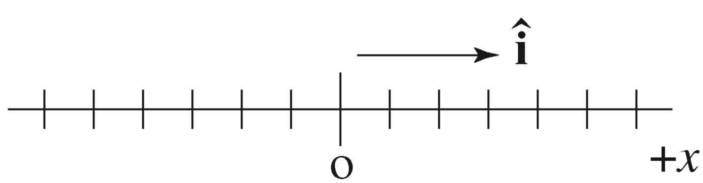


Figure 4.1 A one-dimensional Cartesian coordinate system.

[1] Richard P. Feynman, Robert B. Leighton, Matthew Sands, *The Feynman Lectures on Physics*, Addison-Wesley, Reading, Massachusetts, (1963), p. 12-2.

4.2 Position, Time Interval, Displacement

4.2.1 Position

Consider an object moving in one dimension. We denote the ***position coordinate*** of the center of mass of the object *with respect to the choice of origin* by $x(t)$. The position coordinate is a function of time and can be positive, zero, or negative, depending on the location of the object. The position has both direction and magnitude, and hence is a vector (Figure 4.2),

$$\vec{\mathbf{x}}(t) = x(t)\,\hat{\mathbf{i}}\,. \tag{4.2.1}$$

We denote the position coordinate of the center of the mass at $t = 0$ by the symbol $x_0 \equiv x(t = 0)$. The SI unit for position is the meter [m].

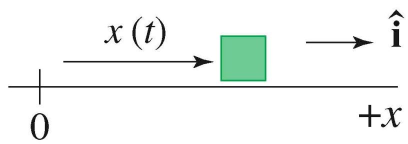


Figure 4.2 The position vector, with reference to a chosen origin.

4.2.2 Time Interval

Consider a closed interval of time $[t_1, t_2]$. We characterize this time interval by the difference in endpoints of the interval such that

$$\Delta t = t_2 - t_1\,. \tag{4.2.2}$$

The SI units for time intervals are seconds [s].

4.2.3 Displacement

The ***displacement***, of a body between times t_1 and t_2 (Figure 4.3) is defined to be the change in position coordinate of the body

$$\Delta\vec{\mathbf{x}} \equiv (x(t_2) - x(t_1))\,\hat{\mathbf{i}} \equiv \Delta x(t)\,\hat{\mathbf{i}}\,. \tag{4.2.3}$$

Displacement is a vector quantity.

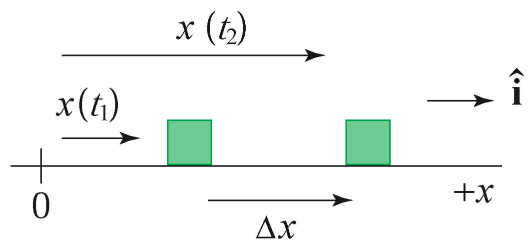


Figure 4.3 The displacement vector of an object over a time interval is the vector difference between the two position vectors

4.3 Velocity

When describing the motion of objects, words like "speed" and "velocity" are used in common language; however when introducing a mathematical description of motion, we need to define these terms precisely. Our procedure will be to define average quantities for finite intervals of time and then examine what happens in the limit as the time interval becomes infinitesimally small. This will lead us to the mathematical concept that velocity at an instant in time is the derivative of the position with respect to time.

4.3.1 Average Velocity

The component of the ***average velocity***, $\overline{v_x}$, for a time interval Δt is defined to be the displacement Δx divided by the time interval Δt,

$$\overline{v_x} \equiv \frac{\Delta x}{\Delta t}. \tag{4.3.1}$$

The average velocity vector is then

$$\overline{\vec{\mathbf{v}}}(t) \equiv \frac{\Delta x}{\Delta t}\,\hat{\mathbf{i}} = \overline{v_x}(t)\,\hat{\mathbf{i}}\,. \tag{4.3.2}$$

The SI units for average velocity are meters per second $\left[\mathrm{m}\cdot\mathrm{s}^{-1}\right]$.

4.3.3 Instantaneous Velocity

Consider a body moving in one direction. We denote the position coordinate of the body by $x(t)$, with initial position x_0 at time $t = 0$. Consider the time interval $[t, t + \Delta t]$. The average velocity for the interval Δt is the slope of the line connecting the points $(t, x(t))$ and $(t + \Delta t, x(t + \Delta t))$. The slope, the rise over the run, is the change in position over the change in time, and is given by

$$\overline{v}_x \equiv \frac{\text{rise}}{\text{run}} = \frac{\Delta x}{\Delta t} = \frac{x(t+\Delta t) - x(t)}{\Delta t}. \tag{4.3.3}$$

Let's see what happens to the average velocity as we shrink the size of the time interval. The slope of the line connecting the points $(t, x(t))$ and $(t+\Delta t, x(t+\Delta t))$ approaches the slope of the tangent line to the curve $x(t)$ at the time t (Figure 4.4).

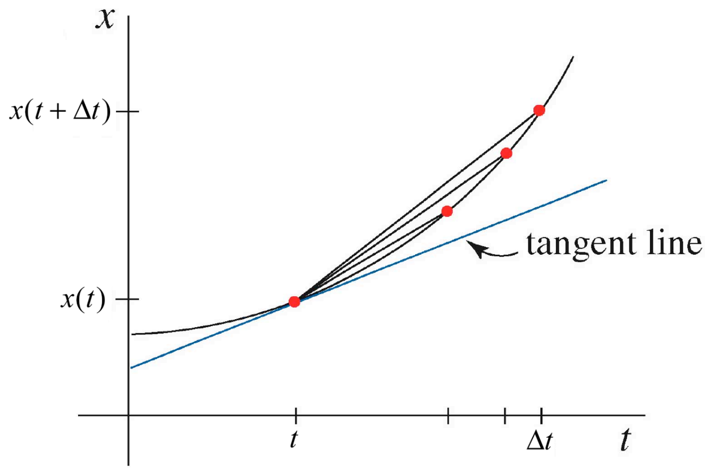


Figure 4.4 Graph of position *vs*. time showing the tangent line at time t.

In order to define the limiting value for the slope at any time, we choose a time interval $[t, t+\Delta t]$. For each value of Δt, we calculate the average velocity. As $\Delta t \to 0$, we generate a sequence of average velocities. The limiting value of this sequence is defined to be the x-component of the instantaneous velocity at the time t.

The x-component of ***instantaneous velocity*** at time t is given by the slope of the tangent line to the curve of position vs. time at time t:

$$v_x(t) \equiv \lim_{\Delta t \to 0} \overline{v}_x = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} \equiv \frac{dx}{dt}. \tag{4.3.4}$$

The instantaneous velocity vector is then

$$\vec{\mathbf{v}}(t) = v_x(t)\,\hat{\mathbf{i}}. \tag{4.3.5}$$

Example 4.1 Determining Velocity from Position

Consider an object that is moving along the x-coordinate axis represented by the equation

$$x(t) = x_0 + \frac{1}{2}bt^2 \tag{4.3.6}$$

where x_0 is the initial position of the object at $t = 0$.

We can explicitly calculate the x-component of instantaneous velocity from Equation (4.3.4) by first calculating the displacement in the x-direction, $\Delta x = x(t+\Delta t) - x(t)$. We need to calculate the position at time $t + \Delta t$,

$$x(t+\Delta t) = x_0 + \frac{1}{2}b(t+\Delta t)^2 = x_0 + \frac{1}{2}b\left(t^2 + 2t\Delta t + \Delta t^2\right). \tag{4.3.7}$$

Then the instantaneous velocity is

$$v_x(t) = \lim_{\Delta t \to 0}\frac{x(t+\Delta t) - x(t)}{\Delta t} = \lim_{\Delta t \to 0}\frac{\left(x_0 + \frac{1}{2}b(t^2 + 2t\,\Delta t + \Delta t^2)\right) - \left(x_0 + \frac{1}{2}b\,t^2\right)}{\Delta t}. \tag{4.3.8}$$

This expression reduces to

$$v_x(t) = \lim_{\Delta t \to 0}\left(b\,t + \frac{1}{2}b\,\Delta t\right). \tag{4.3.9}$$

The first term is independent of the interval Δt and the second term vanishes because the limit as $\Delta t \to 0$ of Δt is zero. Thus the instantaneous velocity at time *t* is

$$v_x(t) = b\,t\,. \tag{4.3.10}$$

In Figure 4.5 we graph the instantaneous velocity, $v_x(t)$, as a function of time t.

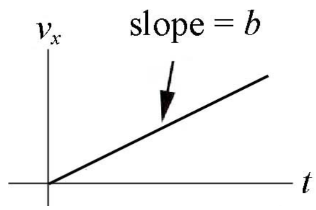


Figure 4.5 A graph of instantaneous velocity as a function of time.

4.4 Acceleration

We shall apply the same physical and mathematical procedure for defining acceleration, the rate of change of velocity. We first consider how the instantaneous velocity changes over an interval of time and then take the limit as the time interval approaches zero.

4.4.1 Average Acceleration

Acceleration is the quantity that measures a change in velocity over a particular time interval. Suppose during a time interval Δt a body undergoes a change in velocity

$$\Delta\vec{\mathbf{v}} = \vec{\mathbf{v}}(t+\Delta t) - \vec{\mathbf{v}}(t) . \quad (4.4.1)$$

The change in the x-component of the velocity, Δv_x, for the time interval $[t, t+\Delta t]$ is then

$$\Delta v_x = v_x(t+\Delta t) - v_x(t) . \quad (4.4.2)$$

The x-***component of the average acceleration*** for the time interval Δt is defined to be

$$\vec{\overline{\mathbf{a}}} = \overline{a_x}\,\hat{\mathbf{i}} \equiv \frac{\Delta v_x}{\Delta t}\,\hat{\mathbf{i}} = \frac{(v_x(t+\Delta t) - v_x(t))}{\Delta t}\,\hat{\mathbf{i}} = \frac{\Delta v_x}{\Delta t}\,\hat{\mathbf{i}} . \quad (4.4.3)$$

The SI units for average acceleration are meters per second squared, $[\mathrm{m \cdot s^{-2}}]$.

4.4.2 Instantaneous Acceleration

On a graph of the x-component of velocity *vs*. time, the average acceleration for a time interval Δt is the slope of the straight line connecting the two points $(t, v_x(t))$ and $(t+\Delta t, v_x(t+\Delta t))$. In order to define the x-component of the instantaneous acceleration at time t, we employ the same limiting argument as we did when we defined the instantaneous velocity in terms of the slope of the tangent line.

The x-***component of the instantaneous acceleration*** *at time* t is the limit of the slope of the tangent line at time t of the graph of the x-component of the velocity as a function of time,

$$a_x(t) \equiv \lim_{\Delta t \to 0} \overline{a_x} = \lim_{\Delta t \to 0} \frac{(v_x(t+\Delta t) - v_x(t))}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta v_x}{\Delta t} \equiv \frac{dv_x}{dt} . \quad (4.4.4)$$

The instantaneous acceleration vector is then

$$\vec{\mathbf{a}}(t) = a_x(t)\,\hat{\mathbf{i}} . \quad (4.4.5)$$

In Figure 4.6 we illustrate this geometrical construction. Because the velocity is the derivative of position with respect to time, the x-component of the acceleration is the second derivative of the position function,

$$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2} . \quad (4.4.6)$$

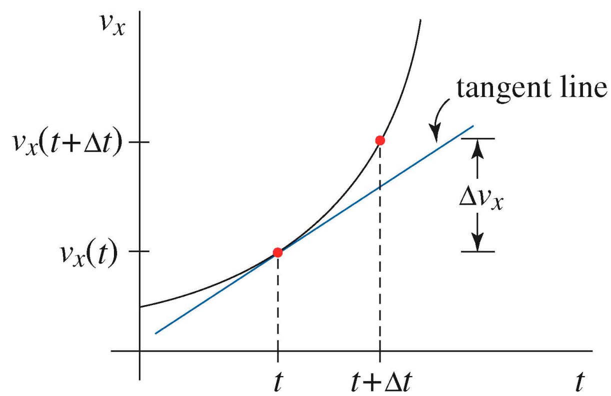


Figure 4.6 Graph of velocity *vs*. time showing the tangent line at time t.

Example 4.2 Determining Acceleration from Velocity

Let's continue Example 4.1, in which the position function for the body is given by $x = x_0 + (1/2)bt^2$, and the x-component of the velocity is $v_x = bt$. The x-component of the instantaneous acceleration at time t is the limit of the slope of the tangent line at time t of the graph of the x-component of the velocity as a function of time (Figure 4.5)

$$a_x = \frac{dv_x}{dt} = \lim_{\Delta t \to 0} \frac{v_x(t+\Delta t) - v_x(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{bt + b\Delta t - bt}{\Delta t} = b. \qquad (4.4.7)$$

Note that in Equation (4.4.7), the ratio $\Delta v / \Delta t$ is independent of t, consistent with the constant slope of the graph in Figure 4.5.

4.5 Constant Acceleration

Let's consider a body undergoing constant acceleration for a time interval $\Delta t = [0, t]$. When the acceleration a_x is a constant, the average acceleration is equal to the instantaneous acceleration. Denote the x-component of the velocity at time $t = 0$ by $v_{x,0} \equiv v_x(t = 0)$. Therefore the x-component of the acceleration is given by

$$a_x = \overline{a_x} = \frac{\Delta v_x}{\Delta t} = \frac{v_x(t) - v_{x,0}}{t}. \qquad (4.5.1)$$

Thus the velocity as a function of time is given by

$$v_x(t) = v_{x,0} + a_x t. \qquad (4.5.2)$$

When the acceleration is constant, the velocity is a linear function of time.

4.5.1 Velocity: Area Under the Acceleration *vs*. Time Graph

In Figure 4.7, the x-component of the acceleration is graphed as a function of time.

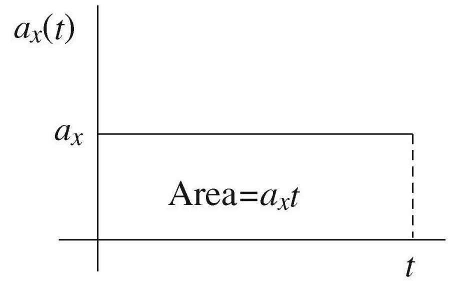


Figure 4.7 Graph of the x-component of the acceleration for a_x constant as a function of time.

The area under the acceleration *vs*. time graph, for the time interval $\Delta t = t - 0 = t$, is

$$\text{Area}(a_x, t) \equiv a_x\, t\,. \tag{4.5.3}$$

Using the definition of average acceleration given above,

$$\text{Area}(a_x, t) \equiv a_x\, t = \Delta v_x = v_x(t) - v_{x,0}\,. \tag{4.5.4}$$

4.5.2 Displacement: Area Under the Velocity *vs*. Time Graph

In Figure 4.8, we graph the x-component of the velocity *vs*. time curve.

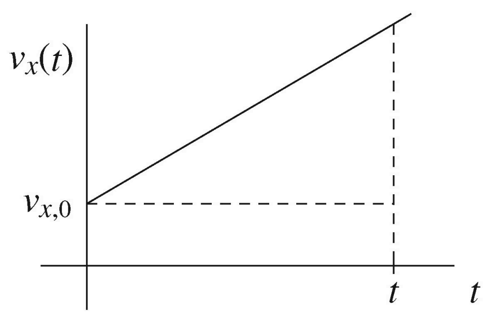


Figure 4.8 Graph of velocity as a function of time for a_x constant.

The region under the velocity *vs*. time curve is a trapezoid, formed from a rectangle and a triangle and the area of the trapezoid is given by

$$\text{Area}(v_x, t) = v_{x,0}\, t + \frac{1}{2}(v_x(t) - v_{x,0})t\,. \tag{4.5.5}$$

Substituting for the velocity (Equation (4.5.2)) yields

$$\text{Area}(v_x, t) = v_{x,0}\, t + \frac{1}{2}(v_{x,0} + a_x\, t - v_{x,0})t = v_{x,0}\, t + \frac{1}{2} a_x\, t^2 . \quad (4.5.6)$$

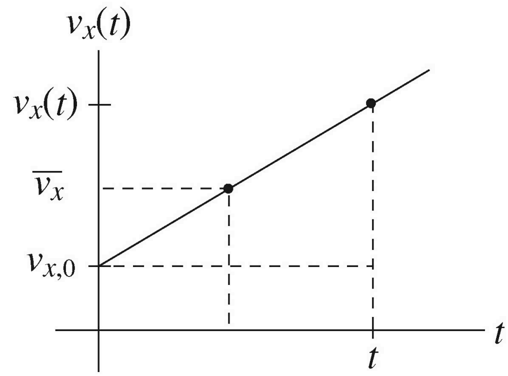


Figure 4.9 The average velocity over a time interval.

We can then determine the average velocity by adding the initial and final velocities and dividing by a factor of two (Figure 4.9).

$$\overline{v_x} = \frac{1}{2}(v_x(t) + v_{x,0}) . \quad (4.5.7)$$

The above method for determining the average velocity differs from the definition of average velocity in Equation (4.3.1). When the acceleration is constant over a time interval, the two methods will give identical results. Substitute into Equation (4.5.7) the x-component of the velocity, Equation (4.5.2), to yield

$$\overline{v_x} = \frac{1}{2}(v_x(t) + v_{x,0}) = \frac{1}{2}((v_{x,0} + a_x\, t) + v_{x,0}) = v_{x,0} + \frac{1}{2} a_x\, t . \quad (4.5.8)$$

Recall Equation (4.3.1); the average velocity is the displacement divided by the time interval (note we are now using the definition of average velocity that always holds, for non-constant as well as constant acceleration). The displacement is equal to

$$\Delta x \equiv x(t) - x_0 = \overline{v_x}\, t . \quad (4.5.9)$$

Substituting Equation (4.5.8) into Equation (4.5.9) shows that displacement is given by

$$\Delta x \equiv x(t) - x_0 = \overline{v_x}\, t = v_{x,0}\, t + \frac{1}{2} a_x\, t^2 . \quad (4.5.10)$$

Now compare Equation (4.5.10) to Equation (4.5.6) to conclude that the displacement is equal to the area under the graph of the x-component of the velocity *vs*. time,

$$\Delta x \equiv x(t) - x_0 = v_{x,0}\, t + \frac{1}{2} a_x\, t^2 = \text{Area}(v_x, t), \tag{4.5.11}$$

and so we can solve Equation (4.5.11) for the position as a function of time,

$$x(t) = x_0 + v_{x,0}\, t + \frac{1}{2} a_x\, t^2 . \tag{4.5.12}$$

Figure 4.10 shows a graph of this equation. Notice that at $t = 0$ the slope may be in general non-zero, corresponding to the initial velocity component $v_{x,0}$.

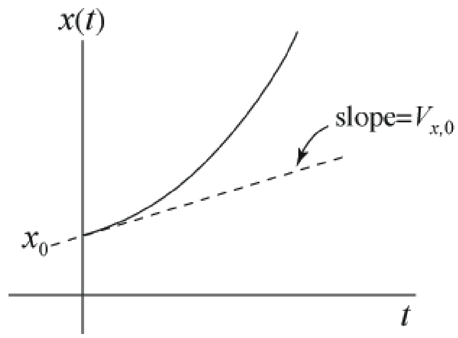


Figure 4.10 Graph of position vs. time for constant acceleration.

Example 4.3 Accelerating Car

A car, starting at rest at $t = 0$, accelerates in a straight line for 100 m with an unknown constant acceleration. It reaches a speed of $20\text{ m}\cdot\text{s}^{-1}$ and then continues at this speed for another 10 s. (a) Write down the equations for position and velocity of the car as a function of time. (b) How long was the car accelerating? (c) What was the magnitude of the acceleration? (d) Plot speed *vs.* time, acceleration *vs.* time, and position *vs.* time for the entire motion. (e) What was the average velocity for the entire trip?

Solutions: (a) For the acceleration a, the position $x(t)$ and velocity $v(t)$ as a function of time t for a car starting from rest are

$$\begin{aligned} x(t) &= (1/2)\,a\,t^2 \\ v_x(t) &= a\,t. \end{aligned} \tag{4.5.13}$$

b) Denote the time interval during which the car accelerated by t_1. We know that the position $x(t_1) = 100\text{m}$ and $v(t_1) = 20\text{ m}\cdot\text{s}^{-1}$. Note that we can eliminate the acceleration a between the Equations (4.5.13) to obtain

$$x(t) = (1/2)\,v(t)\; t . \tag{4.5.14}$$

We can solve this equation for time as a function of the distance and the final speed giving

$$t = 2\frac{x(t)}{v(t)}. \quad (4.5.15)$$

We can now substitute our known values for the position $x(t_1) = 100\,\text{m}$ and $v(t_1) = 20\ \text{m}\cdot\text{s}^{-1}$ and solve for the time interval that the car has accelerated

$$t_1 = 2\frac{x(t_1)}{v(t_1)} = 2\frac{100\ \text{m}}{20\ \text{m}\cdot\text{s}^{-1}} = 10\,\text{s}. \quad (4.5.16)$$

c) We can substitute into either of the expressions in Equation (4.5.13); the second is slightly easier to use,

$$a = \frac{v(t_1)}{t_1} = \frac{20\ \text{m}\cdot\text{s}^{-1}}{10\,\text{s}} = 2.0\,\text{m}\cdot\text{s}^{-2}. \quad (4.5.17)$$

d) The x-component of acceleration vs. time, x-component of the velocity vs. time, and the position vs. time are piece-wise functions given by

$$a_x(t) = \begin{cases} 2\ \text{m}\cdot\text{s}^{-2}; & 0 < t < 10\ \text{s} \\ 0; & 10\ \text{s} < t < 20\ \text{s} \end{cases},$$

$$v_x(t) = \begin{cases} (2\ \text{m}\cdot\text{s}^{-2})t; & 0 < t < 10\ \text{s} \\ 20\ \text{m}\cdot\text{s}^{-1}; & 10\ \text{s} < t < 20\ \text{s} \end{cases},$$

$$x(t) = \begin{cases} (1/2)(2\ \text{m}\cdot\text{s}^{-2})t^2; & 0 < t < 10\ \text{s} \\ 100\ \text{m} + (20\ \text{m}\cdot\text{s}^{-2})(t - 10\ \text{s}); & 10\ \text{s} < t < 20\ \text{s} \end{cases}.$$

The graphs of the x-component of acceleration vs. time, x-component of the velocity vs. time, and the position vs. time are shown in Figure 4.11

(e) After accelerating, the car travels for an additional ten seconds at constant speed and during this interval the car travels an additional distance $\Delta x = v(t_1) \times 10\,\text{s} = 200\,\text{m}$ (note that this is twice the distance traveled during the $10\,\text{s}$ of acceleration), so the total distance traveled is $300\,\text{m}$ and the total time is $20\,\text{s}$, for an average velocity of

$$v_{\text{ave}} = \frac{300\,\text{m}}{20\,\text{s}} = 15\,\text{m}\cdot\text{s}^{-1}. \quad (4.5.18)$$

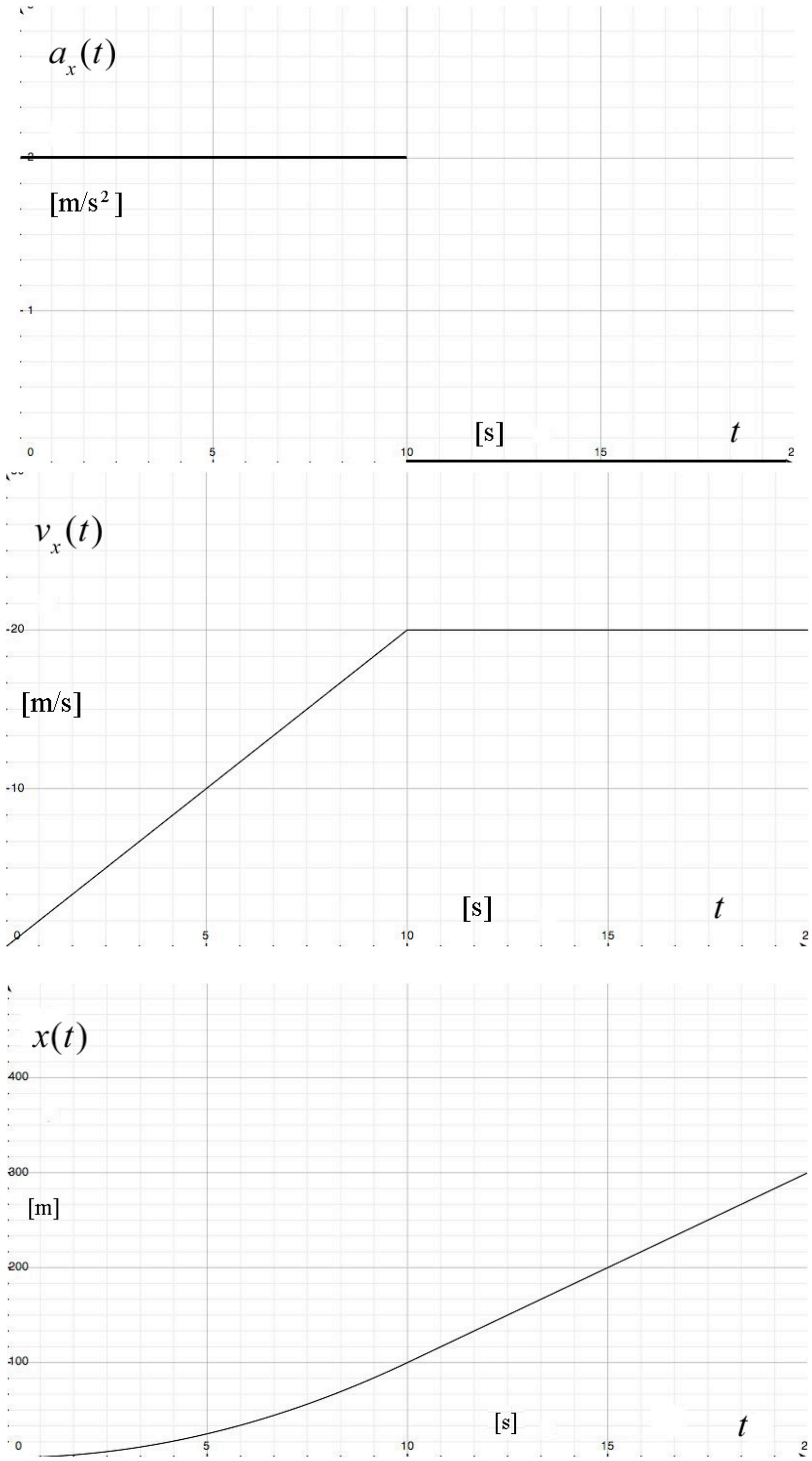


Figure 4.11 Graphs of the x-components of acceleration, velocity and position as piece-wise functions of time

Example 4.4 Catching a Bus

At the instant a traffic light turns green, a car starts from rest with a given constant acceleration, $5.0\times10^{-1}\,\mathrm{m}\cdot\mathrm{s}^{-2}$. Just as the light turns green, a bus, traveling with a given constant speed, $1.6\times10^{1}\,\mathrm{m}\cdot\mathrm{s}^{-1}$, passes the car. The car speeds up and passes the bus some time later. How far down the road has the car traveled, when the car passes the bus?

Solution: In this example we will illustrate the Polya approach to problem solving.

1. Understand – get a conceptual grasp of the problem

Think about the problem. How many objects are involved in this problem? Two, the bus and the car. How many different stages of motion are there for each object? Each object has one stage of motion. For each object, how many independent directions are needed to describe the motion of that object? We need only one independent direction for each object. What information can you infer from the problem? The acceleration of the car, the velocity of the bus, and that the position of the car and the bus are identical when the bus just passes the car. Sketch qualitatively the position of the car and bus as a function of time (Figure 4.12).

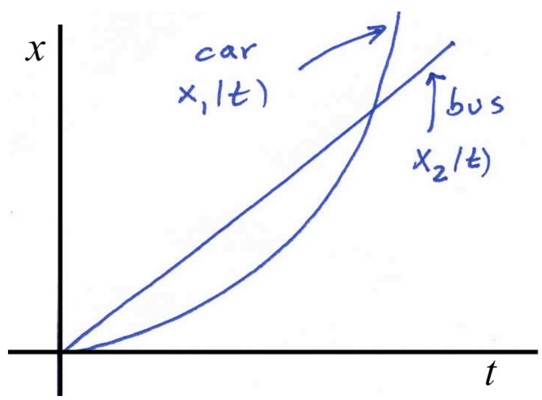


Figure 4.12 Position vs. time of the car and bus.

What choice of coordinate system best suits the problem? Cartesian coordinates with a choice of coordinate system in which the car and bus begin at the origin and travel along the positive x-axis (Figure 4.13). Draw arrows for the position coordinate function for the car and bus.

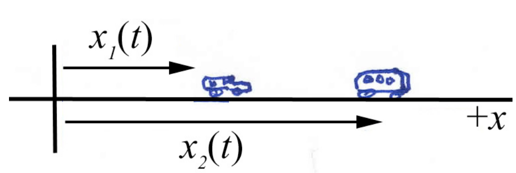


Figure 4.13 A coordinate system for car and bus.

2. Devise a Plan - set up a procedure to obtain the desired solution

Write down the complete set of equations for the position and velocity functions. There are two objects, the car and the bus. Choose a coordinate system with the origin at the traffic light with the car and bus traveling in the positive x-direction. Call the position function of the car, $x_1(t)$, and the position function for the bus, $x_2(t)$. In general the position and velocity functions of the car are given by

$$x_1(t) = x_1(0) + v_{x_1}(0)t + \frac{1}{2}a_{x_1}t^2 ,$$
$$v_{x_1}(t) = v_{x_1}(0) + a_{x_1}t .$$

In this example, using both the information from the problem and our choice of coordinate system, the initial position and initial velocity of the car are both zero, $x_1(0) = 0$ and $v_{x_1}(0) = 0$, and the acceleration of the car is non-zero $a_{x_1} \neq 0$. So the position and velocity of the car is given by

$$x_1(t) = \frac{1}{2}a_{x_1}t^2 ,$$
$$v_{x_1}(t) = a_{x_1}t .$$

The initial position of the bus is zero, $x_2(0) = 0$, the initial velocity of the bus is non-zero, $v_{x_2}(0) \neq 0$, and the acceleration of the bus is zero, $a_{x_2} = 0$. Therefore the velocity is constant, $v_{x_2}(t) = v_{x_2}(0)$, and the position function for the bus is given by $x_2(t) = v_{x_2}(0)t$.

Identify any specified quantities. The problem states: "The car speeds up and passes the bus some time later." What analytic condition best expresses this condition? Let $t = t_a$ correspond to the time that the car passes the bus. Then at that time, the position functions of the bus and car are equal, $x_1(t_a) = x_2(t_a)$.

How many quantities need to be specified in order to find a solution? There are three independent equations at time $t = t_a$: the equations for position and velocity of the car $x_1(t_a) = \frac{1}{2}a_{x_1}t_a^{\,2}$, $v_{x_1}(t_a) = a_{x_1}t_a$, and the equation for the position of the bus, $x_2(t) = v_{x_2}(0)t$. There is one 'constraint condition' $x_1(t_a) = x_2(t_a)$.

The six quantities that are as yet unspecified are $x_1(t_a)$, $x_2(t_a)$, $v_{x_1}(t_a)$, $v_{x_2}(0)$, a_{x_1}, t_a. Therefore you need to be given at least two numerical values in order to completely specify all the quantities; for example the distance where the car and bus meet. The problem specifying the initial velocity of the bus, $v_{x_2}(0)$, and the acceleration, a_{x_1}, of the car with given values.

3. Carry our your plan – solve the problem!

The number of independent equations is equal to the number of unknowns so you can design a strategy for solving the system of equations for the distance the car has traveled in terms of the velocity of the bus $v_{x_2}(0)$ and the acceleration of the car a_{x_1}, when the car passes the bus.

Let's use the constraint condition to solve for the time $t = t_a$ where the car and bus meet. Then we can use either of the position functions to find out where this occurs. Thus the constraint condition, $x_1(t_a) = x_2(t_a)$ becomes $(1/2)a_{x_1} t_a^{\,2} = v_{x_2}(0)t_a$. We can solve for this time, $t_a = 2v_{x_2}(0)/a_{x_1}$. Therefore the position of the car at the meeting point is

$$x_1(t_a) = \frac{1}{2}a_{x_1} t_a^{\,2} = \frac{1}{2}a_{x,1}\left(2\frac{v_{x_2}(0)}{a_{x_1}}\right)^2 = \frac{2v_{x_2}(0)^2}{a_{x_1}}.$$

4. Look Back – check your solution and method of solution

Check your algebra. Do your units agree? The units look good since in the answer the two sides agree in units, $\left[\mathrm{m}\right] = \left[\mathrm{m}^2 \cdot \mathrm{s}^{-2} / \mathrm{m} \cdot \mathrm{s}^{-2}\right]$ and the algebra checks. Substitute in numbers. Suppose $a_{x_1} = 5.0 \times 10^{-1}\,\mathrm{m} \cdot \mathrm{s}^{-2}$ and $v_{x_2}(0) = 1.6 \times 10^{1}\,\mathrm{m} \cdot \mathrm{s}^{-1}$, Introduce your numerical values for $v_{x_2}(0)$ and a_{x_1}, and solve numerically for the distance the car has traveled when the bus just passes the car. Then

$$t_a = \frac{2v_{x_2}(0)}{a_{x_1}} = \frac{(2)\left(1.6 \times 10^{1}\,\mathrm{m} \cdot \mathrm{s}^{-1}\right)}{\left(5.0 \times 10^{-1}\,\mathrm{m} \cdot \mathrm{s}^{-2}\right)} = 6.4 \times 10^{1}\,\mathrm{s},$$

$$x_1(t_a) = \frac{2v_{x_2}(0)^2}{a_{xL1}} = \frac{(2)\left(1.6 \times 10^{1}\,\mathrm{m} \cdot \mathrm{s}^{-1}\right)^2}{\left(5.0 \times 10^{-1}\,\mathrm{m} \cdot \mathrm{s}^{-2}\right)} = 1.0 \times 10^{3}\,\mathrm{m}.$$

Check your results. Once you have an answer, think about whether it agrees with your estimate of what it should be. Some very careless errors can be caught at this point. Is it possible that when the car just passes the bus, the car and bus have the same velocity? Then there would be an additional constraint condition at time $t = t_a$, that the velocities are equal, $v_{x,1}(t_a) = v_{x,20}$. Thus $v_{x,1}(t_a) = a_{x,1}t_a = v_{x,20}$ implies that $t_a = v_{x,20} / a_{x,1}$. From our other result for the time of intersection $t_a = 2v_{x_2}(0) / a_{x_1}$. But these two results contradict each other, so it is not possible.

4.6 One Dimensional Kinematics Non-Constant Acceleration

4.6.1 Change of Velocity as the Integral of Non-constant Acceleration

When the acceleration is a non-constant function, we would like to know how the x-component of the velocity changes for a time interval $\Delta t = [0, t]$. Since the acceleration is non-constant we cannot simply multiply the acceleration by the time interval. We shall calculate the change in the x-component of the velocity for a small time interval $\Delta t_i \equiv [t_i, t_{i+1}]$ and sum over these results. We then take the limit as the time intervals become very small and the summation becomes an integral of the x-component of the acceleration.

For a time interval $\Delta t = [0, t]$, we divide the interval up into N small intervals $\Delta t_i \equiv [t_i, t_{i+1}]$, where the index $i = 1, 2, \ldots, N$, and $t_1 \equiv 0$, $t_{N+1} \equiv t$. Over the interval Δt_i, we can approximate the acceleration as a constant, $\overline{a_x(t_i)}$. Then the change in the x-component of the velocity is the area under the acceleration *vs*. time curve,

$$\Delta v_{x,i} \equiv v_x(t_{i+1}) - v_x(t_i) = \overline{a_x(t_i)}\, \Delta t_i + E_i, \tag{4.6.1}$$

where E_i is the error term (see Figure 4.14a). Then the sum of the changes in the x-component of the velocity is

$$\sum_{i=1}^{i=N} \Delta v_{x,i} = (v_x(t_2) - v_x(t_1 = 0)) + (v_x(t_3) - v_x(t_2)) + \cdots + (v_x(t_{N+1} = t) - v_x(t_N)). \tag{4.6.2}$$

In this summation pairs of terms of the form $\left(v_x(t_2) - v_x(t_2)\right) = 0$ sum to zero, and the overall sum becomes

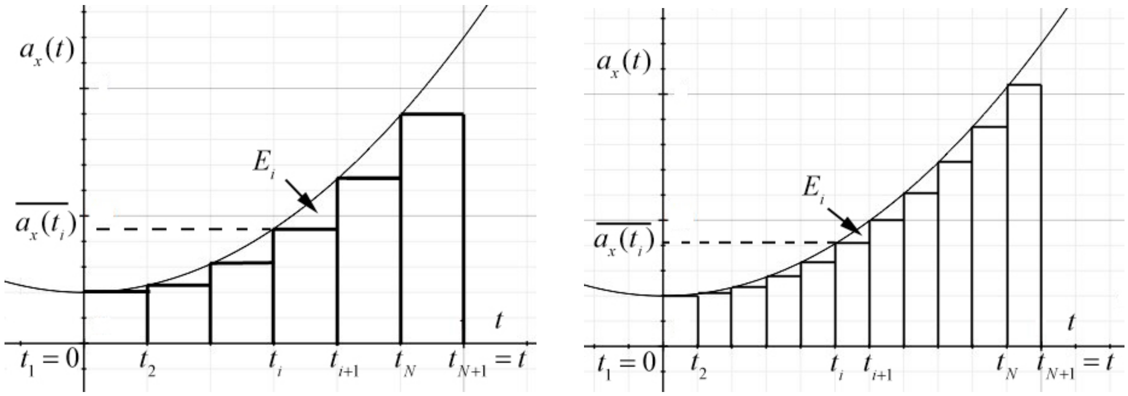
$$v_x(t) - v_x(0) = \sum_{i=1}^{i=N} \Delta v_{x,i}\,. \tag{4.6.3}$$

Substituting Equation (4.6.1) into Equation (4.6.3),

$$v_x(t) - v_x(0) = \sum_{i=1}^{i=N} \Delta v_{x,i} = \sum_{i=1}^{i=N} \overline{a_x(t_i)}\ \Delta t_i + \sum_{i=1}^{i=N} E_i\,. \tag{4.6.4}$$

We now approximate the area under the graph in Figure 4.14a by summing up all the rectangular area terms,

$$\text{Area}_N(a_x,t) = \sum_{i=1}^{i=N} \overline{a_x(t_i)}\ \Delta t_i\ . \tag{4.6.5}$$



Figures 4.14a and 4.14b Approximating the area under the graph of the x-component of the acceleration *vs*. time

Suppose we make a finer subdivision of the time interval $\Delta t = [0, t]$ by increasing N, as shown in Figure 4.14b. The error in the approximation of the area decreases. We now take the limit as N approaches infinity and the size of each interval Δt_i approaches zero. For each value of N, the summation in Equation (4.6.5) gives a value for $\text{Area}_N(a_x,t)$, and we generate a sequence of values

$$\{\text{Area}_1(a_x,t), \text{Area}_2(a_x,t), ..., \text{Area}_N(a_x,t)\}\ . \tag{4.6.6}$$

The limit of this sequence is the area, $\text{Area}(a_x,t)$, under the graph of the x-component of the acceleration *vs*. time. When taking the limit, the error term vanishes in Equation (4.6.4),

$$\lim_{N\to\infty} \sum_{i=1}^{i=N} E_i = 0\ . \tag{4.6.7}$$

Therefore in the limit as N approaches infinity, Equation (4.6.4) becomes

$$v_x(t) - v_x(0) = \lim_{N\to\infty} \sum_{i=1}^{i=N} \overline{a_x(t_i)}\ \Delta t_i + \lim_{N\to\infty} \sum_{i=1}^{i=N} E_i = \lim_{N\to\infty} \sum_{i=1}^{i=N} \overline{a_x(t_i)}\ \Delta t_i = \text{Area}(a_x,t)\ , \tag{4.6.8}$$

and thus the change in the x-component of the velocity is equal to the area under the graph of x-component of the acceleration *vs*. time.

The ***integral of the x-component of the acceleration*** for the interval $[0, t]$ is defined to be the limit of the sequence of areas, $\text{Area}_N(a_x,t)$, and is denoted by

$$\int_{t'=0}^{t'=t} a_x(t')\,dt' \equiv \lim_{\Delta t_i \to 0} \sum_{i=1}^{i=N} a_x(t_i)\,\Delta t_i = \text{Area}(a_x, t)\,. \qquad (4.6.9)$$

Equation (4.6.8) shows that the change in the x–component of the velocity is the integral of the x-component of the acceleration with respect to time.

$$v_x(t) - v_x(0) = \int_{t'=0}^{t'=t} a_x(t')\,dt'\,. \qquad (4.6.10)$$

Using integration techniques, we can in principle find the expressions for the velocity as a function of time for any acceleration.

4.6.2 Integral of Velocity

We can repeat the same argument for approximating the area $\text{Area}(v_x, t)$ under the graph of the x-component of the velocity *vs*. time by subdividing the time interval into N intervals and approximating the area by

$$\text{Area}_N(a_x, t) = \sum_{i=1}^{i=N} \overline{v_x(t_i)}\;\Delta t_i\,. \qquad (4.6.11)$$

The displacement for a time interval $\Delta t = [0, t]$ is limit of the sequence of sums $\text{Area}_N(a_x, t)$,

$$\Delta x = x(t) - x(0) = \lim_{N \to \infty} \sum_{i=1}^{i=N} \overline{v_x(t_i)}\;\Delta t_i\,. \qquad (4.6.12)$$

This approximation is shown in Figure 4.15.

The ***integral of the*** x ***-component of the velocity*** for the interval $[0, t]$ is the limit of the sequence of areas, $\text{Area}_N(a_x, t)$, and is denoted by

$$\int_{t'=0}^{t'=t} v_x(t')\,dt' \equiv \lim_{\Delta t_i \to 0} \sum_{i=1}^{i=N} v_x(t_i)\,\Delta t_i = \text{Area}(v_x, t)\,. \qquad (4.6.13)$$

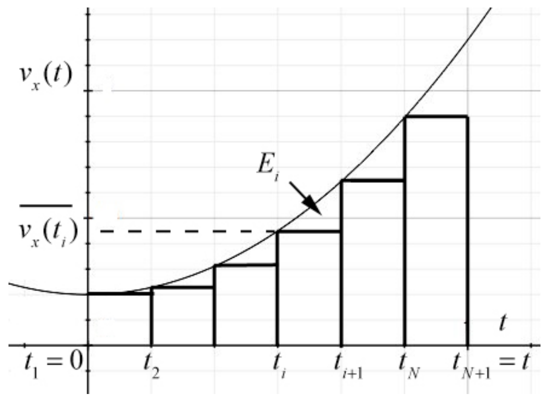


Figure 4.15 Approximating the area under the graph of the x-component of the velocity *vs*. time.

The displacement is then the integral of the x-component of the velocity with respect to time,

$$\Delta x = x(t) - x(0) = \int_{t'=0}^{t'=t} v_x(t')\,dt' . \tag{4.6.14}$$

Using integration techniques, we can in principle find the expressions for the position as a function of time for any acceleration.

Example 4.5 Non-constant Acceleration

Let's consider a case in which the acceleration, $a_x(t)$, is not constant in time,

$$a_x(t) = b_0 + b_1 t + b_2 t^2 . \tag{4.6.15}$$

The graph of the x-component of the acceleration *vs*. time is shown in Figure 4.16

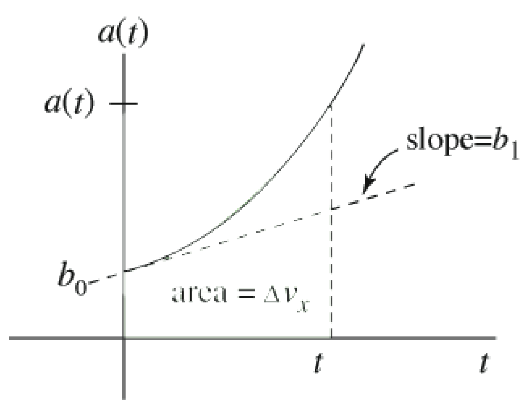


Figure 4.16 Non-constant acceleration *vs*. time graph.

Let's find the change in the x-component of the velocity as a function of time. Denote the initial velocity at $t = 0$ by $v_{x,0} \equiv v_x(t=0)$. Then,

$$v_x(t) - v_{x,0} = \int_{t'=0}^{t'=t} a_x(t')\,dt' = \int_{t'=0}^{t'=t} (b_o + b_1 t' + b_2 t'^2)\,dt' = b_0 t + \frac{b_1 t^2}{2} + \frac{b_2 t^3}{3}. \quad (4.6.16)$$

The x-component of the velocity as a function in time is then

$$v_x(t) = v_{x,0} + b_0 t + \frac{b_1 t^2}{2} + \frac{b_2 t^3}{3}. \quad (4.6.17)$$

Denote the initial position by $x_0 \equiv x(t=0)$. The displacement as a function of time is the integral

$$x(t) - x_0 = \int_{t'=0}^{t'=t} v_x(t')\,dt'. \quad (4.6.18)$$

Use Equation (4.6.17) for the x-component of the velocity in Equation (4.6.18) to find

$$x(t) - x_0 = \int_{t'=0}^{t'=t} \left(v_{x,0} + b_0 t' + \frac{b_1 t'^2}{2} + \frac{b_2 t'^3}{3} \right) dt' = v_{x,0}\, t + \frac{b_0 t^2}{2} + \frac{b_1 t^3}{6} + \frac{b_2 t^4}{12}. \quad (4.6.19)$$

Finally the position is then

$$x(t) = x_0 + v_{x,0}\, t + \frac{b_0 t^2}{2} + \frac{b_1 t^3}{6} + \frac{b_2 t^4}{12}. \quad (4.6.20)$$

Example 4.6 Bicycle and Car

A car is driving through a green light at $t = 0$ located at $x = 0$ with an initial speed $v_{c,0} = 12\ \text{m}\cdot\text{s}^{-1}$. The acceleration of the car as a function of time is given by

$$a_c = \begin{cases} 0; & 0 < t < t_1 = 1\ \text{s} \\ -(6\ \text{m}\cdot\text{s}^{-3})(t - t_1); & 1\ \text{s} < t < t_2 \end{cases}.$$

(a) Find the speed and position of the car as a function of time. (b) A bicycle rider is riding at a constant speed of $v_{b,0}$ and at $t = 0$ is $17\ \text{m}$ behind the car. The bicyclist reaches the car when the car just comes to rest. Find the speed of the bicycle.

Solution: a) We need to integrate the acceleration for both intervals. The first interval is easy, the speed is constant. For the second integral we need to be careful about the endpoints of the integral and the fact that the integral is the change in speed so we must subtract $v_c(t_1) = v_{c0}$

$$v_c(t) = \begin{cases} v_{c0}; & 0 < t < t_1 = 1\text{ s} \\ v_c(t_1) + \int\limits_{t'=t_1}^{t'=t} -(6\text{ m}\cdot\text{s}^{-3})(t'-t_1); & 1\text{ s} < t < t_2 \end{cases}.$$

After integrating we get

$$v_c(t) = \begin{cases} v_{c0}; & 0 < t < t_1 = 1\text{ s} \\ v_{c0} - (3\text{ m}\cdot\text{s}^{-3})(t'-t_1)^2\Big|_{t'=t_1}^{t'=t}; & 1\text{ s} < t < t_2 \end{cases}.$$

Now substitute the endpoint so the integral to finally yield

$$v_c(t) = \begin{cases} v_{c0} = 12\text{ m}\cdot\text{s}^{-1}; & 0 < t < t_1 = 1\text{ s} \\ 12\text{ m}\cdot\text{s}^{-1} - (3\text{ m}\cdot\text{s}^{-3})(t-t_1)^2; & 1\text{ s} < t < t_2 \end{cases}.$$

For this one-dimensional motion the change in position is the integral of the speed so

$$x_c(t) = \begin{cases} x_c(0) + \int\limits_0^{t_1} (12\text{ m}\cdot\text{s}^{-1})\,dt; & 0 < t < t_1 = 1\text{ s} \\ x_c(t_1) + \int\limits_{t'=t_1}^{t'=t} \left(12\text{ m}\cdot\text{s}^{-1} - (3\text{m}\cdot\text{s}^{-3})(t'-t_1)^2\right)dt; & 1\text{ s} < t < t_2 \end{cases}.$$

Upon integration we have

$$x_c(t) = \begin{cases} x_c(0) + (12\text{ m}\cdot\text{s}^{-1})t; & 0 < t < t_1 = 1\text{ s} \\ x_c(t_1) + \left((12\text{ m}\cdot\text{s}^{-1})(t'-t_1) - (1\text{ m}\cdot\text{s}^{-3})(t'-t_1)^3\right)\Big|_{t'=t_1}^{t'=t}; & 1\text{ s} < t < t_2 \end{cases}.$$

We choose our coordinate system such that $x_c(0) = 0$, therefore $x_c(t_1) = (12\text{ m}\cdot\text{s}^{-1})(1\text{ s}) = 12\text{ m}$. So after substituting in the endpoints of the integration interval we have that

$$x_c(t) = \begin{cases} (12\text{ m}\cdot\text{s}^{-1})t; & 0 < t < t_1 = 1\text{ s} \\ 12\text{ m} + (12\text{ m}\cdot\text{s}^{-1})(t-t_1) - (1\text{ m}\cdot\text{s}^{-3})(t-t_1)^3; & 1\text{ s} < t < t_2 \end{cases}.$$

(b) We are looking for the instant t_2 that the car has come to rest. So we use our expression for the speed for the interval $1\text{ s} < t < t_2$, $0 = v_c(t_2) = 12\text{ m}\cdot\text{s}^{-1} - (3\text{ m}\cdot\text{s}^{-3})(t_2 - t_1)^2$. We can solve this for t_2: $(t_2 - t_1)^2 = 4\text{ s}^2$. We have two solutions: $(t_2 - t_1) = 2\text{ s}$ or $(t_2 - t_1) = -2\text{ s}$. The second solution $t_2 = t_1 - 2\text{ s} = 1\text{ s} - 2\text{ s} = -1\text{ s}$ does not apply to our time interval and so $t_2 = t_1 + 2\text{ s} = 1\text{ s} + 2\text{ s} = 3\text{ s}$. The position of the car at t_2 is then given by

$$\begin{aligned} x_c(t_2) &= 12\text{ m} + (12\text{ m}\cdot\text{s}^{-1})(t_2 - t_1) - (1\text{ m}\cdot\text{s}^{-3})(t_2 - t_1)^3 \\ &= 12\text{ m} + (12\text{ m}\cdot\text{s}^{-1})(2\text{ s}) - (1\text{ m}\cdot\text{s}^{-3})(2\text{ s})^3 = 28\text{ m}. \end{aligned}$$

Because the bicycle is traveling at a constant speed with an initial position $x_{b0} = -17\text{ m}$, the position of the bicycle is given by $x_b(t) = -17\text{ m} + v_b t$. The bicycle and car intersect at instant $t_2 = 3\text{ s}$: $x_b(t_2) = x_c(t_2)$. Therefore $-17\text{ m} + v_b(3\text{ s}) = 28\text{ m}$. So the speed of the bicycle is

$$v_b = \frac{(28\text{ m} + 17\text{ m})}{(3\text{ s})} = 15\text{ m}\cdot\text{s}^{-1}.$$